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ON THE STABILITY OF A PLAIN HOLLOW VORTEX

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Proof is given of the idifferent stability with respect to small perturbations of two flows: a hollow vortex bounded on the outside by a circular wall, and a free hollow vortex.

A method of analyzing the stability of plane potential flows of a perfect incompressible fluid with respect to small perturbations was suggested in [1] by which the difficulties arising in the determination of eigenfunctions of two-dimensional hydrodynamic flows. The method proposed there for the analysis of stability consists of the linearization of equations of hydrodynamics by conformal mapping of the unperturbed flow region onto that of the perturbed flow. It is applicable to fairly simple regions of the unperturbed flow, otherwise the feasability of conformal mapping becomes problematic. This aspect was not touched upon in [1]; some of the flows considered by the Authors cannot be analyzed in this way, since for these conformal mapping is impossible. Neither the question of completeness of the system of eigenfunctions in cases in which mapping is possible was investigated by them. It is, therefore, interesting to examine the equations arising in investigations of small perturbations of stationary flows by the method of conformal mapping, to determine its limits of applicability and, also, to solve Cauchy's problem in terms of perturbation equations.

1. A hollow vortex bounded on the outside by a circular wall. The potential flow of an incompressible fluid in the form of a plane hollow vortex bounded on the outside by a circular wall is considered. With the notation $z_0 = x_0 + iy_0$ for the complex variable in the physical plane of flow and $\zeta = u_0 - iv_0$ for the complex velocity, the flow velocity is given by formula

$$\zeta = -i\zeta^{-1}, \quad 1 \leqslant |z_0| \leqslant r^{-1}, \quad 0 < r < 1 \tag{1.1}$$

The flow boundary $|z_0| = 1$ is free and the pressure at it is constant: $p = \text{const}$. The line $|z_0| = r^{-1}$ is the rigid wall and the flow hodograph is represented by the ring

$$r \leqslant |\zeta| \leqslant 1 \tag{1.2}$$

The complex flow potential $f_0 = \varphi_0 + i\psi_0$ is defined by formula

$$f_0 = -i \ \ln \ z_0 \tag{1.3}$$

Let at the initial instant of time $t = 0$ the flow be acted upon by perturbing forces (*) and the perturbed flow potential $f(z_0, \ 0)$ can be expressed by

$$f(z_0, \ 0) = -i \ \ln z_0 + \varepsilon f_1(z_0, \ 0) + O(\varepsilon^2) \tag{1.4}$$

where ε is a small real parameter, and $f_1(z_0, 0)$ is assumed to be analytic and bounded in the region of flow.

Since the perturbation is assumed to be potential, the equations of the perturbed flow are of the form

$$\operatorname{Re}\frac{\partial f}{\partial t} + \frac{w\bar{w}}{2} + \frac{p}{\rho_0} = 0, \qquad w = \frac{\partial f}{\partial z} \tag{1.5}$$

where $f(z, \ t)$ is the complex potential, w is the complex velocity, z is a physical variable in the perturbed flow region, and ρ_0 is the constant density of the fluid.

At the initial instant the velocity field of the perturbed flow differs only slightly from that of the unperturbed flow, hence it can be reasonably assumed that at subsequent instants the perturbed flow region D_t does not appreciably differ from the unperturbed flow region D_0, i.e., the boundary of D_t is of the form

$$1 + \delta(\theta) \leqslant | \ z_0 \ | \leqslant r^{-1}, \qquad \delta \ (\theta) \in C^2 \ [0, \ 2\pi] \tag{1.6}$$

and $\delta(\theta) \sim \varepsilon$, $\delta'(\theta) \sim \varepsilon$ and $\delta''(\theta) \sim \varepsilon$. Since the fluid is assumed incompressible, the areas of D_0 and D_t are equal, i.e.,

$$\int_0^{2\pi} \delta(\theta)\, d\theta = 0 \tag{1.7}$$

Relationship (1.7) makes it possible to assert that the conformal mapping $D_0 \to D_t$ exists in the first approximation, hence function

$$z = z(z_0, \ \bar{z}_0, \ t) = z_0 + \varepsilon z_1(z_0, \ t) + O(\varepsilon^2) \tag{1.8}$$

is analytic to within terms of the order of ε^2.

When the form of the free boundary $\delta(\theta)$ is known, it is possible to derive the explicit expression for $z_1(z_0, t)$.

Let us actually seek $z(z_0, \bar{z}_0, t)$ in the form $z_0 + \Delta z$. In the first approximation we have

$$\operatorname{Re}(\Delta z \bar{z}_0) = \begin{cases} 0, & |z_0| = r^{-1} \\ \delta(\theta), & |z_0| = 1 \end{cases}$$

If $\delta(\theta)$ is specified by a Fourier series with coefficients a_k, then

$$\Delta z = \sum_{k=-\infty,\, k\neq 1}^{k=+\infty} 2a_{k-1} z_0^k \left(r^{k+1} - r^{3-k}\right)^{-1}$$

Using (1.8) we seek the unknown functions in the form

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*) Physically perturbations can be induced by a momentary pulse of force or by a perturbation of the free surface.

$$\begin{aligned} p(z, t) &= p_0(z_0) + \varepsilon p_1(z_0, t) \\ w(z,t) &= \zeta(z_0) + \varepsilon w_1(z_0, t) \\ f(z, t) &= f_0(z_0) + \varepsilon[f_1(z_0, t) + \zeta z_1(z_0, t)] \end{aligned} \tag{1.9}$$

and linearize (1.5)

$$p_1 = -\rho_0 \operatorname{Re}\left(\frac{\partial f_1}{\partial t} + \bar{\zeta} w_1\right), \quad w_1 = -i\zeta^2\left(\frac{\partial f_1}{\partial \zeta} + z_1\right) \tag{1.10}$$

These equations, which are valid throughout region D_0, must be supplemented by relationships satisfied at the free boundary of the perturbed flow

$$p = \text{const}, \qquad (dz / dt)_n = (\bar{w})_n \tag{1.11}$$

Linearizing relationships (1.11) by means of (1.9) and using (1.10), we obtain

$$\operatorname{Re}\left(\frac{\partial f_1}{\partial t} + \frac{w_1}{\zeta}\right) = 0, \quad \operatorname{Im}\left(\zeta\frac{\partial z_1}{\partial t} - i\zeta^2\frac{\partial z_1}{\partial \zeta} + \frac{w_1}{\zeta}\right) = 0 \tag{1.12}$$

Relationships (1.12) are satisfied at the perturbed flow boundary, since at the circumference by virtue of (1.8) $|\zeta| = 1$. Relationships (1.12) must be supplemented by conditions at the wall

$$\operatorname{Im} f_1 = 0, \qquad \operatorname{Im}(\zeta z_1) = 0 \quad \text{for} \quad |\zeta| = r \tag{1.13}$$

It follows from (1.13) that the second of relationships (1.12) is valid for $|\zeta| = r$, hence

$$\zeta\frac{\partial z_1}{\partial t} - i\zeta^2\frac{\partial z_1}{\partial \zeta} + \frac{w_1}{\zeta} = b(t) \tag{1.14}$$

where $b(t)$ is a real function of time. Owing to the arbitrariness of the right-hand side of (1.14), the latter admits the particular solution

$$z_1 = \frac{1}{\zeta}\int_0^t b(\tau)\, d\tau$$

which represents a rotation of the whole region D_0 around $z_0 = 0$. Its occurrence is due to the mapping $D_0 \to D_t$ being defined in (1.8) to within the rotation of D_0. Hence for normalizing the mapping we set $b(t) = 0$, and from (1.10), (1.12) and (1.14) we then obtain

$$\operatorname{Im}\left(i\frac{\partial^2 f_1}{\partial t^2} + 2\zeta\frac{\partial^2 f_1}{\partial t \partial \zeta} - i\zeta^2\frac{\partial^2 f_1}{\partial \zeta^2}\right) = 0, \quad \text{for } |\zeta| = 1 \tag{1.15}$$

The latter condition together with relationships (1.12) and the relationship

$$\operatorname{Im} f_1 = 0 \quad \text{for } |\zeta| = r \tag{1.16}$$

completely defines at $|\zeta| = 1$ the solution of the problem.

Let at the initial instant $t = 0$ functions $f_1(\zeta, 0)$ and $z_1(\zeta, 0)$, specified by their expansions into Laurent series in the hodograph region (1.12) be given by

$$\begin{aligned} f_1(\zeta, 0) &= \sum_{n\neq 0} s_n\zeta^n, \qquad s_{-n} = \bar{s}_n r^{2n} \\ z_1(\zeta, 0) &= \sum_{n\neq 0} q_n\zeta^{n-1}, \qquad q_{-n} = \bar{q}_n r^{2n} \end{aligned} \tag{1.17}$$

Initial perturbations satisfy conditions (1.13) and (1.7). We seek the solution of the problem in the form of a Laurent expansion in the ring (1.2). By satisfying for $b(t) = 0$

conditions (1.12), (1.15) - (1.17), and (1.14) we obtain

$$\begin{pmatrix} f_1(\zeta, t) \\ z_1(\zeta, t) \end{pmatrix} = \sum_{n\neq 0} \left\{ P_n \begin{pmatrix} \zeta^n \\ \varkappa_n^{-1}\zeta^{n-1} \end{pmatrix} e^{\lambda_n t} + Q_n \begin{pmatrix} \zeta^n \\ -\varkappa_n^{-1}\zeta^{n-1} \end{pmatrix} e^{\mu_n t} \right\} \quad (1.18)$$

$$P_n = {}^1/_2 (s_n + q_n\varkappa_n), \qquad Q_n = {}^1/_2 (s_n - q_n\varkappa_n)$$

$$\lambda_n = in(1 + \varkappa_n), \qquad \mu_n = in(1 - \varkappa_n)$$

$$\varkappa_n = n^{-1/2}\left(\frac{1 - r^{2n}}{1 + r^{2n}}\right)^{1/2}$$

The derived formulas make it possible to resolve the question of the smoothness of input data necessary for the convergence of series (1.18) and for the existence and continuity at the boundaries of D_0 of derivatives

$$dz_1/dt,\ d^2f_1/dt^2,\ dz_1/d\zeta,\ d^2f_1/d\zeta^2$$

and for the dislocation δ (θ) to have a small curvature. In fact, it is sufficient that $f_1(z_0, 0) \in C^4(D_0)$ and $z_1(z_0, 0) \in C^3(D_0)$. Using Privalov's lemma [2], we can reduce the smoothness requirements for the input data, i.e., it is possible to show that stipulation for

$$\frac{\partial^3 f_1(e^{i\theta}, 0)}{\partial\theta^3}, \qquad \frac{\partial^2 z_1(e^{i\theta}, 0)}{\partial\theta^2}$$

to uniformly satisfy the Hölder condition at the circumference $|z_0| = 1$ is sufficient.

The series expansion (1.18) provides the solution of the problem of stability with respect to small perturbations of the flow considered above. The stability of this flow is indifferent and has a discrete spectrum of purely imaginary eigenvalues λ_n and μ_n. The system of eigenfunctions (in parentheses in the right-hand side of (1.18)) is complete in the space of input data (1.17).

The flow stability is related to the stabilizing effect of the centrifugal force which at the free boundary removes the linear increase of perturbation with respect to time. In fact, with a circular wall inside and a free boundary outside of the vortex ($r > 1$) the flow becomes absolutely unstable.

2. A free hollow vortex. Let us consider the limit case of the flow described in Sect. 1 for infinitely distant wall, i.e., for $r \to 0$. Region D_0 is then outside the unit circle $|z_0| \geqslant 1$, and the hodograph region is the unit circle with point $\zeta = 0$ removed. Function $f_1(z_0, 0)$ is assumed bounded at an infinitely distant point and function $z_1(z_0, 0)$ is supposed to satisfy condition (1.7), which is equivalent to the requirement that $p_1(z_0, 0) \to 0$ for $|z_0| \to \infty$.

In this case the conformal mapping $D_0 \to D_t$ is feasible (not necessarily in the first approximation), and perturbation equations are particularly simple. Equation (1.15) then extends to the inside of the unit circle (since the functions appearing in it are bounded) and takes the form

$$\frac{\partial^2 f_1}{\partial t^2} - 2i\frac{\partial^2 f_1}{\partial t\partial\zeta} + \zeta^2\frac{\partial^2 f_1}{\partial\zeta^2} = 0 \quad \text{for } |\zeta| \leqslant 1 \quad (2.1)$$

From relationships (1.12) we find that

$$\partial f_1/\partial t = -w_1\zeta^{-1} \quad \text{for } |\zeta| \leqslant 1$$

Hence

$$\left.\frac{\partial f_1}{\partial t}\right|_{t=0} = i\zeta\left(\frac{\partial f_1(\zeta,0)}{\partial \zeta} + z_1(\zeta,0)\right)$$

With known $f_1(\zeta, 0)$ and $\partial f_1(\zeta, 0)/\partial t$ we can find $f_1(\zeta, t)$ from Eq. (2.1) and then integrating (1.14) with $b(t) = 0$, to determine $z_1(\zeta, t)$. In this case the unknown functions are sought in the form of Taylor's series. Formulas defining the solution of this problem are obtained by transition to the limit $r \to 0$ in (1.18).

It should be noted that all statements derived in Sect. 1 are valid in this limit case.

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ON ONE EXACT SOLUTION OF EQUATIONS OF PLAIN NONSTATIONARY MOTION OF GAS

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An exact solution of equations of a plain nonstationary potential isentropic motion of gas dependent on two arbitrary functions with the Poisson adiabatic exponent equal to two is derived. The solution is interpreted as the motion of "shallow water" with a free surface which must be ruled. The general aspects of shallow water motion, and in particular the case of a cylindrical free surface an nonunivariate motion are considered.

1. The equation defining a plane nonstationary potential isentropic motion of gas is taken in the form [1]

$$\begin{gathered}(\Phi_H)^2 + \Phi_{uu}\Phi_{vv} - (\Phi_{uv})^2 - \Phi_H(\Phi_{uu} + \Phi_{vv}) + \\ + (\gamma - 1) H \left[(\Phi_{uH})^2 + (\Phi_{vH})^2 + 2\Phi_H\Phi_{HH} - \Phi_{HH}(\Phi_{uu} + \Phi_{vv})\right] = 0\end{gathered} \tag{1.1}$$

where u and v are projections of the velocity vector $\mathbf{v}$ on the x and y axes of a Cartesian system of coordinates, H is the enthalpy, t is the time, γ is the Poisson adiabatic exponent, and Φ is the conjugate potential related to the velocity potential φ by formula

$$\Phi = \varphi - xu - yv + t\left[{}^1/_2(u^2 + v^2) + H\right] \tag{1.2}$$

Transition to variables t, x, y is by formulas